PHYSICAL REVIEW E **71**, 055303(R) (2005)

Quasiplanar steep water waves

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(Received 28 February 2005; published 24 May 2005)

A unique description for highly nonlinear potential water waves is suggested, where weak three-dimensional effects are included as small corrections to exact two-dimensional equations written in conformal variables. Contrary to the traditional approach, a small parameter in this theory is not a surface slope, but it is the ratio of a typical wavelength to a large transversal scale along the second horizontal coordinate. A first-order correction for the Hamiltonian functional is calculated, and the corresponding equations of motion are derived for steep water waves over an arbitrary nonuniform quasi-one-dimensional bottom profile.

DOI: 10.1103/PhysRevE.71.055303 PACS number(s): 47.15.Hg, 47.35.+i, 47.10.+g

The problem of water waves is one of the classical fields in hydrodynamics, and it has been studied extensively over many years. Starting from the middle of the 1990's, in the theory of two-dimensional (2D) potential flows of an ideal fluid with a free surface, the so-called conformal variables have been employed [1–4] (see also Ref. [5] for surface gravity-capillary waves under external pressure forcing). With these variables, highly nonlinear equations of motion for planar water waves can be represented in an exact and compact form containing integral operators diagonal in the Fourier representation. Such integrodifferential equations are very suitable for numerical simulations, since effective computer programs for the discrete fast Fourier transform (FFT) are now available (see, e.g., [6]). Based on these equations, significant progress has been achieved in understanding nonlinear dynamics of water waves, including the wave turbulence [7,8] and the mechanism of sudden formation of the giant sea waves [9]. Recently, the exact 2D description has been generalized to arbitrary nonuniform space- and time-dependent bottom profiles [10,11]. However, though in many situations real sea waves differ from planar flows just weakly, but they are never ideally two-dimensional, and in natural conditions the second horizontal dimension plays an important role in wave dynamics. Therefore there is a need for a theory, which could describe strongly nonlinear, perhaps breaking waves and, on the other hand, it would take into account three-dimensional (3D) effects, at least as weak corrections to a dominant 2D motion. In the present work such a highly nonlinear weakly 3D theory is suggested as an extension of the exact conformal 2D theory described in [10]. It should be emphasized that existing approximate nonlinear evolution equations for water waves (for example, the famous Kadomtsev-Petviashvili equation, various Boussinesq-type equations [12,13], or the equations derived by Matsuno [14] and Choi [15]) are valid just for weakly nonlinear water waves, but not for overturning or breaking waves.

It is a well-known fact that a very significant difficulty in the 3D theory of potential water waves is the general impossibility to solve the Laplace equation for the velocity potential $\varphi(x,y,q,t)$,

$$\varphi_{xx}+\varphi_{yy}+\varphi_{qq}=0, \tag{1}$$

in the flow region $-H(x,q)\leqslant y\leqslant \eta(x,q,t)$ between a (static for simplicity) bottom and a time-dependent free surface, with the given boundary conditions

$$\varphi|_{y=\eta(x,q,t)}=\psi(x,q,t), \quad (\partial\varphi/\partial n)|_{y=-H(x,q)}=0. \tag{2}$$

(Here x and q are the horizontal Cartesian coordinates, y is the vertical coordinate, while the symbol z will be used for the complex combination $z=x+iy$). Therefore, a compact expression is absent for the Hamiltonian functional of the system,

$$\mathcal{H}\{\eta,\psi\}=\frac{1}{2}\int dxdq\int_{-H(x,q)}^{\eta(x,q,t)}(\varphi_x^2+\varphi_y^2+\varphi_q^2)dy+\frac{g}{2}\int\eta^2dxdq$$
$$\equiv\mathcal{K}\{\eta,\psi\}+\mathcal{P}\{\eta\} \tag{3}$$

(the sum of the kinetic energy of the fluid and the potential energy in the vertical gravitational field g). The Hamiltonian determines canonical equations of motion (see [16–18], and references therein)

$$\eta_t=(\delta\mathcal{H}/\delta\psi), \quad -\psi_t=(\delta\mathcal{H}/\delta\eta), \tag{4}$$

in accordance with the variational principle $\delta\int\tilde{\mathcal{L}}dt=0$, where the Lagrangian is $\tilde{\mathcal{L}}=\int\psi\eta_t dxdq-\mathcal{H}$.

In the traditional approach, the problem is partly solved by an asymptotic expansion of the kinetic energy $\mathcal{K}$ on a small parameter—the steepness of the surface (see Refs. [16,18], and references therein). As a result, a weakly nonlinear theory is generated, which is not good to describe large-amplitude steep waves (see Ref. [19] for a discussion about the limits of such a theory). The theory developed in the present work is based on another small parameter—the ratio of a typical length of the waves propagating along the x axis, to a large scale along the transversal horizontal direction, denoted by q [alternatively, it is the ratio of typical wave numbers k_q/k_x in the Fourier plane (k_x,k_q)]. Thus, we define $\epsilon=(l_x/l_q)^2\ll 1$ and note: the less this parameter, the less our flow differs from a purely 2D flow. A profile $y=\eta(x,q,t)$ of the free surface, a boundary value of the velocity potential $\psi(x,q,t)\equiv\varphi(x,\eta(x,q,t),q,t)$, and a given bottom profile $y=-H(x,q)$ are allowed to depend strongly on

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the coordinate x, while the derivatives over the coordinate q will be supposedly small: $|\eta_q|\sim\epsilon^{1/2}$, $|\psi_q|\sim\epsilon^{1/2}$, $|H_q|\sim\epsilon^{1/2}$.

General idea of the method. In the same manner as in the exact 2D theory [10,11], instead of the Cartesian coordinates x and y, we use curvilinear conformal coordinates u and v, which make the free surface and the bottom effectively flat: $x+iy\equiv z=z(u+iv,q,t)$, $-\infty<u<+\infty$, $0\leqslant v\leqslant 1$, where $z(w,q,t)$ is an analytical on the complex variable $w\equiv u+iv$ function without any singularities in the flow domain $0\leqslant v\leqslant 1$. Now the bottom corresponds to $v=0$, while on the free surface $v=1$. The boundary value of the velocity potential is $\varphi|_{v=1}\equiv\psi(u,q,t)$. If the bottom is nonuniform, it is convenient to represent the conformal mapping $z(w,q,t)$ as a composition of two conformal mappings $w\mapsto\zeta\mapsto z$, similarly to works [10,11]: $z(w,q,t)=Z(\zeta(w,q,t),q)$. Here an intermediate function $\zeta(w,q,t)$ possesses the property $\mathrm{Im}\,\zeta(u+0i,q,t)=0$, thus resulting in the important relation

$$\zeta(u+i,q,t)\equiv\xi(u,q,t)=(1+i\hat{R})\rho(u,q,t), \quad (5)$$

where $\rho(u,q,t)$ is a purely real function, and $\hat{R}=i\tanh\hat{k}$ (here $\hat{k}\equiv -i\hat{\partial}_u$) is an anti-Hermitian operator, which is diagonal in the Fourier representation: it multiplies the Fourier harmonics $\rho_k(q,t)\equiv\int\rho(u,q,t)e^{-iku}du$ by $R_k=i\tanh k$, so that $\hat{R}\rho(u,q,t)=\int[i\tanh k]\rho_k(q,t)e^{iku}(dk/2\pi)$. A known analytical function $Z(\zeta,q)$ determines parametrically the static bottom profile: $X^{[b]}(r,q)+iY^{[b]}(r,q)=Z(r,q)$, where r is a real parameter running from $-\infty$ to $+\infty$. The profile of the free surface is now given (in a parametric form as well) by the formula

$$X^{[s]}(u,q,t)+iY^{[s]}(u,q,t)\equiv Z^{[s]}(u,q,t)=Z(\xi(u,q,t),q). \quad (6)$$

For equations to be shorter, below we do not indicate the arguments (u,q,t) of the functions ψ, ξ $\bar{\xi}$ (the overline denotes complex conjugate). Also, we introduce the notation $Z'(\xi)\equiv\partial_\xi Z(\xi,q)$. The Lagrangian of the system in terms of the variables ψ, ξ, and $\bar{\xi}$ can be written as follows (this is a generalization of the 2D Lagrangian used in [10], to 3D space):

$$\mathcal{L}=\int Z'(\xi)\overline{Z'(\xi)}\left[\frac{\xi_t\bar{\xi}_u-\bar{\xi}_t\xi_u}{2i}\right]\psi du dq-\mathcal{K}\{\psi,Z(\xi),\overline{Z(\xi)}\}$$
$$-\frac{g}{2}\int\left[\frac{Z(\xi)-\overline{Z(\xi)}}{2i}\right]^2\left[\frac{Z'(\xi)\xi_u+\overline{Z'(\xi)}\bar{\xi}_u}{2}\right]du dq$$
$$+\int\Lambda\left[\frac{\xi-\bar{\xi}}{2i}-\hat{R}\left(\frac{\xi+\bar{\xi}}{2}\right)\right]du dq, \quad (7)$$

where the indefinite real Lagrangian multiplier $\Lambda(u,q,t)$ has been introduced in order to take into account the relation (5). Equations of motion follow from the variational principle $\delta\mathcal{A}=0$, with the action $\mathcal{A}\equiv\int\mathcal{L}dt$. So, variation by $\delta\psi$ gives us the first equation of motion—the kinematic condition on the free surface

$$|Z'(\xi)|^2\,\mathrm{Im}(\xi_t\bar{\xi}_u)=(\delta\mathcal{K}/\delta\psi). \quad (8)$$

Let us divide this equation by $|Z'(\xi)|^2|\xi_u|^2$ and use analytical properties of the function ξ_t/ξ_u. As a result, we obtain the time-derivative-resolved equation

$$\xi_t=\xi_u(\hat{T}+i)\left[\frac{(\delta\mathcal{K}/\delta\psi)}{|Z'(\xi)|^2|\xi_u|^2}\right], \quad (9)$$

where the linear operator $\hat{T}\equiv\hat{R}^{-1}=-i\coth\hat{k}$ has been introduced. Further, variation of the action $\mathcal{A}$ by $\delta\xi$ gives us the second equation of motion

$$\left[\frac{\psi_u\bar{\xi}_t-\psi_t\bar{\xi}_u}{2i}\right]|Z'(\xi)|^2=\left(\frac{\delta\mathcal{K}}{\delta Z}\right)Z'(\xi)+\frac{g}{2i}\,\mathrm{Im}(Z(\xi))|Z'(\xi)|^2\bar{\xi}_u$$
$$-\frac{(1+i\hat{R})\Lambda}{2i}. \quad (10)$$

After multiplying Eq. (10) by $-2i\xi_u$ we have

$$\{[\psi_t+g\,\mathrm{Im}\,Z(\xi)]|\xi_u|^2-\psi_u\bar{\xi}_t\xi_u\}|Z'(\xi)|^2$$
$$=(1+i\hat{R})\tilde{\Lambda}-2i\left(\frac{\delta\mathcal{K}}{\delta Z}\right)Z'(\xi)\xi_u, \quad (11)$$

where $\tilde{\Lambda}$ is another real function. Taking the imaginary part of Eq. (11) and using Eq. (8), we find $\tilde{\Lambda}$,

$$\tilde{\Lambda}=\hat{T}\left[\psi_u\frac{\delta\mathcal{K}}{\delta\psi}\right]+2\hat{T}\,\mathrm{Re}\left[\left(\frac{\delta\mathcal{K}}{\delta Z}\right)Z'(\xi)\xi_u\right].$$

After that, the real part of Eq. (11) gives us the Bernoulli equation in a general form,

$$\psi_t+g\,\mathrm{Im}\;Z(\xi)=\psi_u\hat{T}\left[\frac{(\delta\mathcal{K}/\delta\psi)}{|Z'(\xi)|^2|\xi_u|^2}\right]+\frac{\hat{T}[\psi_u(\delta\mathcal{K}/\delta\psi)]}{|Z'(\xi)|^2|\xi_u|^2}$$
$$+\frac{2\,\mathrm{Re}((\hat{T}-i)[(\delta\mathcal{K}/\delta Z)Z'(\xi)\xi_u])}{|Z'(\xi)|^2|\xi_u|^2}. \quad (12)$$

It is interesting to note that equations of motion (9) and (12) possess a definite gauge invariance because they contain only invariant combinations $Z'(\xi)\xi_u\equiv Z_u^{[s]}$ and $Z'(\xi)\xi_t\equiv Z_t^{[s]}$ [obviously, Eq. (9) can be multiplied by $Z'(\xi)$]. Therefore, there is a freedom in choice for the function $Z(\zeta,q)$: instead of a particular function $Z_1(\zeta,q)$, one may use another function $Z_2(\zeta,q)$ if $Z_1(\zeta,q)=Z_2(\tilde{\zeta}(\zeta,q),q)$, where $\tilde{\zeta}(\zeta,q)$ is a sufficiently smooth analytical on ζ function taking real values at the real axis $\mathrm{Im}\,\zeta=0$. Accordingly, $\xi_2(u,q,t)=\tilde{\zeta}(\xi_1(u,q,t),q)$, thus the real functions $\rho_1(u,q,t)$ and $\rho_2(u,q,t)$ are different in both cases. This gauge freedom can be useful in numerical simulations, when a nontrivial bottom topography has to be taken into account.

Equations (9) and (12) completely determine the evolution of the system, provided the kinetic-energy functional $\mathcal{K}\{\psi,Z,\bar{Z}\}$ is explicitly given. It should be emphasized that in our description a general expression for $\mathcal{K}$ remains unknown. However, under the conditions $|z_q|\ll 1$, $|\varphi_q|\ll 1$, the potential

$\varphi(u,v,q,t)$ is efficiently expanded into a series on the powers of the small parameter ϵ,

$$\varphi=\varphi^{(0)}+\varphi^{(1)}+\varphi^{(2)}+\cdots,\quad \varphi^{(n)}\sim\epsilon^{n}, \tag{13}$$

where $\varphi^{(n+1)}$ can be calculated from $\varphi^{(n)}$, and the zeroth-order term $\varphi^{(0)}=\mathrm{Re}\,\phi(w,q,t)$ is the real part of an easily represented (in integral form) analytical function with the boundary conditions $\mathrm{Re}\,\phi|_{v=1}=\psi(u,q,t)$, $\mathrm{Im}\,\phi|_{v=0}=0$. Correspondingly, the kinetic-energy functional will be written in the form $\mathcal{K}=\mathcal{K}^{(0)}+\mathcal{K}^{(1)}+\cdots$, with $\mathcal{K}^{(n)}\sim\epsilon^{n}$, where $\mathcal{K}^{(0)}\{\psi\}$ is the kinetic energy of a purely 2D flow,

$$\mathcal{K}^{(0)}\{\psi\}=\frac{1}{2}\int[(\varphi_u^{(0)})^2+(\varphi_v^{(0)})^2]dudvdq=-\frac{1}{2}\int\psi\hat{R}\psi_u dudq, \tag{14}$$

and the other terms are corrections due to gradients along q. Now we are going to calculate a first-order correction $\mathcal{K}^{(1)}$.

First-order corrections. As a result of the conformal change of two variables, the kinetic-energy functional is determined by the expression

$$\mathcal{K}=\frac{1}{2}\int[\varphi_u^2+\varphi_v^2+J(\mathbf{Q}\cdot\boldsymbol{\nabla}\varphi)^2]dudvdq, \tag{15}$$

where the Cauchy-Riemann conditions $x_u=y_v$, $x_v=-y_u$ have been taken into account, and the following notations are used:

$$J\equiv|z_u|^2,\quad(\mathbf{Q}\cdot\boldsymbol{\nabla}\varphi)\equiv a\varphi_u+b\varphi_v+\varphi_q,$$

$$a=\frac{x_v y_q-x_q y_v}{J}\sim\epsilon^{1/2},\quad b=\frac{y_u x_q-y_q x_u}{J}\sim\epsilon^{1/2}.$$

Consequently, the Laplace equation in the new coordinates takes the form

$$\varphi_{uu}+\varphi_{vv}+\boldsymbol{\nabla}\cdot(\mathbf{Q}J(\mathbf{Q}\cdot\boldsymbol{\nabla}\varphi))=0, \tag{16}$$

with the boundary conditions $\varphi|_{v=1}=\psi(u,q,t)$, and $[\varphi_v+bJ(\varphi_q+a\varphi_u+b\varphi_v)]|_{v=0}=0$. In the limit $\epsilon\ll1$ it is possible to write the solution as the series (13), with the zeroth-order term satisfying the 2D Laplace equation $\varphi_{uu}^{(0)}+\varphi_{vv}^{(0)}=0$, $\varphi|_{v=1}=\psi(u,q,t)$, $\varphi_v|_{v=0}=0$. Thus, it can be represented as $\varphi^{(0)}=\mathrm{Re}\,\phi(w,q,t)$, where

$$\phi(w,q,t)=\int\frac{\psi_k(q,t)e^{ikw}}{\cosh k}\frac{dk}{2\pi}, \tag{17}$$

and $\psi_k(q,t)\equiv\int\psi(u,q,t)e^{-iku}du$. On the free surface

$$\phi(u+i,q,t)\equiv\Psi(u,q,t)=(1+i\hat{R})\psi(u,q,t). \tag{18}$$

For all the other terms in Eq. (13) we have the relations

$$\varphi_{uu}^{(n+1)}+\varphi_{vv}^{(n+1)}+\boldsymbol{\nabla}\cdot(\mathbf{Q}J(\mathbf{Q}\cdot\boldsymbol{\nabla}\varphi^{(n)}))=0 \tag{19}$$

and the boundary conditions

$$\varphi^{(n+1)}|_{v=1}=0,$$

$$[\varphi_v^{(n+1)}+bJ(\varphi_q^{(n)}+a\varphi_u^{(n)}+b\varphi_v^{(n)})]|_{v=0}=0. \tag{20}$$

Noting that $\int(\varphi_u^{(0)}\varphi_u^{(1)}+\varphi_v^{(0)}\varphi_v^{(1)})dudv=0$ (it is easily seen without explicit calculation of $\varphi^{(1)}$ after integration by parts), we have in the first approximation

$$\mathcal{K}^{(1)}=\frac{1}{2}\int J(\varphi_q^{(0)}+a\varphi_u^{(0)}+b\varphi_v^{(0)})^2dudvdq$$
$$=\frac{1}{2}\int z_u\overline{z_u}\left[\mathrm{Re}\left(\phi_q-\frac{\phi_u z_q}{z_u}\right)\right]^2dudvdq. \tag{21}$$

Since $z(w)$ and $\phi(w)$ are represented as $z(u+iv)=e^{\hat{k}(1-v)}\times Z^{[s]}(u)$ and $\phi(u+iv)=e^{\hat{k}(1-v)}\Psi(u)$, we can use for v integration the following auxiliary formulas:

$$\int du\int_0^1[e^{\hat{k}(1-v)}A(u)]\overline{[e^{\hat{k}(1-v)}B(u)]}dv$$
$$=\int\left(\frac{e^{2k}-1}{2k}\right)A_k\overline{B_k}\frac{dk}{2\pi}$$
$$=-\frac{i}{2}\int(\overline{B(u)}\hat{\partial}_u^{-1}A(u)-\overline{B^{[b]}(u)}\hat{\partial}_u^{-1}A^{[b]}(u))du, \tag{22}$$

with $A^{[b]}(u)=e^{\hat{k}}A(u)$, $B^{[b]}(u)=e^{\hat{k}}B(u)$. Now we apply the above formulas to appropriately decomposed Eq. (21) and, as a result, we obtain an expression of a form $\mathcal{K}^{(1)}=\mathcal{K}_{[s]}^{(1)}-\mathcal{K}_{[b]}^{(1)}$, where $\mathcal{K}_{[s]}^{(1)}=\mathcal{F}\{\Psi,\overline{\Psi},Z,\overline{Z}\}$, $\mathcal{K}_{[b]}^{(1)}=\mathcal{F}\{\Psi^{[b]},\overline{\Psi^{[b]}},Z^{[b]},\overline{Z^{[b]}}\}$, with $Z=Z^{[s]}$, $Z^{[b]}=e^{\hat{k}}Z$, $\Psi^{[b]}=e^{\hat{k}}\Psi=[\cosh\hat{k}]^{-1}\psi$. The functional $\mathcal{F}$ is defined as

$$\mathcal{F}=\frac{i}{8}\int(Z_u\Psi_q-Z_q\Psi_u)\partial_u^{-1}\overline{(Z_u\Psi_q-Z_q\Psi_u)}dudq$$
$$+\frac{i}{16}\int\{[(Z_u\Psi_q-Z_q\Psi_u)^2/Z_u]\overline{Z}$$
$$-Z\overline{[(Z_u\Psi_q-Z_q\Psi_u)^2/Z_u]}\}dudq. \tag{23}$$

From here one can express the variational derivatives $(\delta\mathcal{K}^{(1)}/\delta\psi)$ and $(\delta\mathcal{K}^{(1)}/\delta Z)$ by the formulas

$$\frac{\delta\mathcal{K}^{(1)}}{\delta\psi}=\left[(1-i\hat{R})\frac{\delta\mathcal{F}}{\delta\Psi}+(1+i\hat{R})\frac{\delta\mathcal{F}}{\delta\overline{\Psi}}\right]$$
$$-[\cosh\hat{k}]^{-1}\left(\frac{\delta\mathcal{K}_{[b]}^{(1)}}{\delta\Psi^{[b]}}+\frac{\delta\mathcal{K}_{[b]}^{(1)}}{\delta\overline{\Psi^{[b]}}}\right), \tag{24}$$

$$\frac{\delta\mathcal{K}^{(1)}}{\delta Z}=\frac{\delta\mathcal{F}}{\delta Z}-e^{-\hat{k}}\left(\frac{\delta\mathcal{K}_{[b]}^{(1)}}{\delta Z^{[b]}}\right). \tag{25}$$

The derivatives $(\delta\mathcal{F}/\delta\Psi)$ and $(\delta\mathcal{F}/\delta Z)$ are calculated in a standard manner,

$$\frac{\delta\mathcal{F}}{\delta\Psi}=\frac{i}{8}Z_q[\overline{(Z_u\Psi_q-Z_q\Psi_u)}+\hat{\partial}_u[(\Psi_q-Z_q\Psi_u/Z_u)\overline{Z}]]$$
$$-\frac{i}{8}Z_u\hat{\partial}_q[\hat{\partial}_u^{-1}\overline{(Z_u\Psi_q-Z_q\Psi_u)}+(\Psi_q-Z_q\Psi_u/Z_u)\overline{Z}],$$

$$\frac{\delta\mathcal{F}}{\delta Z} = -\frac{i}{8}\Psi_q[\overline{(Z_u\Psi_q - Z_q\Psi_u)} + \hat{\partial}_u[(\Psi_q - Z_q\Psi_u/Z_u)\overline{Z}]] + \frac{i}{8}\Psi_u\hat{\partial}_q[\hat{\partial}_u^{-1}\overline{(Z_u\Psi_q - Z_q\Psi_u)} + (\Psi_q - Z_q\Psi_u/Z_u)\overline{Z}] + \frac{i}{16}[\hat{\partial}_u[(\Psi_q - Z_q\Psi_u/Z_u)^2\overline{Z}] - \overline{(\Psi_q - Z_q\Psi_u/Z_u)^2 Z_u}].$$

The expressions for $(\delta\mathcal{K}^{(1)}_{[b]}/\delta\Psi^{[b]})$ and for $(\delta\mathcal{K}^{(1)}_{[b]}/\delta Z^{[b]})$ are similar. Now one can substitute $(\delta\mathcal{K}/\delta\psi)\approx -\hat{R}\psi_u + (\delta\mathcal{K}^{(1)}/\delta\psi)$ and $(\delta\mathcal{K}/\delta Z)\approx(\delta\mathcal{K}^{(1)}/\delta Z)$ into the equations of motion (9) and (12), keeping in mind that $Z=Z(\xi,q)$, $Z_u = Z'(\xi)\xi_u$, $Z_q = Z'(\xi)\xi_q + \partial_q Z$, $Z^{[b]} = Z([\cosh\hat{k}]^{-1}\rho,q)$, and so on. Thus, the required weakly 3D equations of motion are completely derived, and our main goal is achieved.

The answers are more compact in the limit $|k|\gg 1$, corresponding to the "deep water," when $\hat{R}\to\hat{H}$, $\hat{T}\to -\hat{H}$, with $\hat{H}$ being the Hilbert operator: $\hat{H}=i\,\mathrm{sign}\,\hat{k}$. In this case $\mathcal{K}^{(1)}_{[b]}\to 0$, and therefore

$$\mathcal{K}_{deep} \approx -\frac{1}{2}\int \psi\hat{H}\psi_u du dq + \mathcal{F}\{\Psi,\overline{\Psi},Z,\overline{Z}\}. \quad (26)$$

After appropriate rescaling of the variable u, one may write

$$Z = u + (i-\hat{H})Y(u,q,t), \quad Z_u = 1 + (i-\hat{H})Y_u. \quad (27)$$

Equations of motion for quasiplanar waves on the deep water look as follows:

$$Z_t = Z_u(\hat{H}-i)[[\hat{H}\psi_u - (\delta\mathcal{F}/\delta\psi)]/|Z_u|^2], \quad (28)$$

$$\psi_t + gY = \psi_u\hat{H}[[\hat{H}\psi_u - (\delta\mathcal{F}/\delta\psi)]/|Z_u|^2] + \hat{H}[\psi_u[\hat{H}\psi_u - (\delta\mathcal{F}/\delta\psi)]]/|Z_u|^2 - 2\,\mathrm{Re}((\hat{H}+i)[Z_u(\delta\mathcal{F}/\delta Z)])/|Z_u|^2, \quad (29)$$

where $(\delta\mathcal{F}/\delta\psi) = 2\,\mathrm{Re}[(1-i\hat{H})(\delta\mathcal{F}/\delta\Psi)]$.

In summary, now we have derived fully nonlinear evolution equations for weakly 3D steep water waves, both for the deep water case and for waves over an arbitrary nonuniform quasi one-dimensional bottom profile. A range of possible applications of this theory is very wide. The obtained equations are intended to describe, for example, a sudden formation of giant waves in a sea, or overturning waves on a beach. The next step should be the development of an efficient numerical method for the simulation of these equations, since at the present moment their analytical treatment seems to be very hard. Recently, activity in this direction has been undertaken by the author, and some preliminary numerical results have been already obtained (for deep water waves at this stage, as this case is the most simple for programming). After necessary modifications and careful testing, this code will be used for future serious numerical experiments.

These investigations were supported by RFBR, by the Russian State Program for Support of the Leading Scientific Schools, and by the Russian Science Support Foundation.

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